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A Semigroup Approach to Semilinear Functional Differential Equations in a Banach Space

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Abstract

A class of semilinear autonomous functional differential equations of retarded type is studied by associating with it a evolution equation in the space of initial data, the space C ([-r, 0] ; X). Existence, uniqueness and regularity results a proved.

I. Introduction

Let X be a Banach space endowed with a norm |.| and r be any nonnegative real number.

Let C = C ([-r, 0] ; X) be the Banach space of continuous functions mapping the interval [-r, 0] into X.

Consider the following semilinear functional differential equation of retarded type :

$$\frac{dx}{dt} = Ax(t) + F(x_t), x_0 = \phi = C([-r,0];X), 0 \leq t \leq T; \qquad (1)$$

By a Solution x of equation (1) we mean a function mapping [-r, T] into X, such that $x_t \in$ C ([-r, 0] ; X), For all t ∈ [0, T] and satisfying equation (1), where x_t is the history of x at time t defined pointwise by

$$x_t(\Theta) = x(t+\Theta)\ ;\ -r \leq \theta \leq 0$$

For $\phi \in$ C ([-r, 0] ; X), we set

$$\|\phi\| = sup_{\theta \in [-r,0]} \ |\phi(\theta)|$$

And suppose the following hypotheses

we prove that $\tilde{A}$ generates a semigroup of operators $\tilde{T}$(t) and x(t) defined by :

$$x(t) = \phi\ (t) \text{ if } t \in [-r,0]$$
$$x(t) = (\tilde{T}(t)\phi)(0) \text{ if } t \in [0,+\infty[$$

Is the unique solution of equation (1), continuous on [-r, +∞ [.
The existence of a nonlinear semigroup of operators associated to (1) is deduced from the following general result of M.G.Crandall and T.Liggett [2] :

Theorem 1 *: Let X be a Banach space*

If A be an operator (nonlinear) α-dissipatif ($\alpha \in R$) and

Im(I-λA) $\supset \overline{D(A)}$; for all $\lambda \succ 0$ small enough

Then

$$T(t)x = lim_{n \to +\infty}(I - \frac{t}{n}A)^{-n}x, existsforallx \in \overline{D(A)}. \qquad (3)$$

And the family of applications T (t) : $\overline{D(A)} \to \overline{D(A)}$ is a nonlinear C_0-semigroup.

The Crandall-Liggett theorem is a generalization of the Hille-Yosida-Phillips theorem [12].

In the linear case, we have $A\phi = lim_{t\to 0}\frac{T(t)\phi - \phi}{t}$ and $t \to T(t)$ is a solution of the Cauchy problem, while this result does not hold in general in the nonlinear case. It holds however if X is reflexive ([1], [3]).

Hence, we prove that the operator˜defined by (2) is a infinitesimal generator of $\tilde{T}$ (t).

Lemme 1 *Suppose That (H_1) and (H_2) are satisfied, then, for all $\lambda \in]0,\frac{1}{\alpha}[$ and $\phi \in C$, we have*

$$(\tilde{J}\phi)(0) = J_\lambda[\lambda F(\tilde{J}\phi) + \phi(0)]. \qquad (5)$$

proof : From proposition 1a), we have for all $\psi \in$ C there exists $\phi \in$D($\tilde{A}$) such that (I-λ$\tilde{A}$)$\phi = \psi$, that is,

$$\phi(\theta) = e^{\frac{\theta}{\lambda}}\phi(0) + \frac{1}{\lambda}\int_\theta^0 e^{\frac{\theta - s}{\lambda}}\psi(s)ds, \qquad (6)$$

and $\phi(0) = \psi(0) + \lambda\phi'(0) = \psi\ (0) + \lambda(A\phi(0) + F)\phi))$.
Therefore

$$\phi(0) = (I - \lambda A)^{-1}[\lambda F(e^{\frac{\cdot}{\lambda}}\phi(0) + \frac{1}{\lambda}\int_{\cdot}^0 e^{\frac{\cdot - s}{\lambda}}\psi(s)ds) + \psi(0)]. \qquad (7)$$

For all $\lambda \in$] 0, $\frac{1}{\alpha}$[and $\phi \in$ C, $\tilde{J}_\lambda\phi$ exists and $\tilde{J}_\lambda\phi \in$ D($\tilde{A}$), given by

$$(\tilde{J}_\lambda\phi)(\theta) = e^{\frac{\theta}{\lambda}}(\tilde{J}_\lambda\phi)(0) + \frac{1}{\lambda}\int_\theta^0 e^{\frac{\theta - s}{\lambda}}\phi(s)ds, for \theta \in [-r,0]. \qquad (8)$$

From (7) and (8), we have $(\tilde{J}_\lambda\phi)(\theta) = e^{\frac{\theta}{\lambda}} J_\lambda[\lambda F(\tilde{J}\phi) + \phi(0)] + \frac{1}{\lambda}\int_\theta^0 e^{\frac{\theta - s}{\lambda}}\phi(s)ds$; so,

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$(\tilde{J}\phi)(0) = J_\lambda[\lambda F(\tilde{J}\phi) + \phi(0)]$.

The nonlinear semigroups of Crandall-Liggett are not always differentiable (see [2]).Then we define the infinitesimal generator of $\tilde{T}(t)$ by $B\phi = \lim_{t\to 0}\frac{\tilde{T}(t)\phi-\phi}{t}$ if $\phi \in D(B) = \left\{\phi \in C : \lim_{t\to 0}\frac{\tilde{T}(t)\phi-\phi}{t} exists\right\}$.

Lemme 2 *Suppose That (H_1) and (H_2) are satisfied, then,*

(a) for all $\phi \in C$, for all $n \geq 1$,we have

$\lim_{t\to 0}\frac{1}{n}\sum_{i=0}^{n-1} J_{\frac{t}{n}}^{(i+1)}[F(\tilde{J}_{\frac{t}{n}}^{n-i}\phi)$ - $F(\phi)] = 0$

(b) for all $\phi \in C$, and $n \geq 1$, we have $\lim_{t\to 0}\frac{1}{n}\sum_{i=0}^{n-1}[J_{\frac{t}{n}}^{(i+1)}$ - $I]$ $F(\phi)] = 0$.

(c) for all $\phi \in C$, such that $\phi(0) \in D(A)$, we have

$\lim_{t\to 0}\lim_{n\to+\infty}\frac{1}{t}$ [$[J_{\frac{t}{n}}^{n}$ - I - $tA]\phi(0) = 0$.

Proposition 2 *Suppose That (H_1) and (H_2) are satisfied, then,*

(a) for all $\phi \in C$, $(\tilde{T}(t)\phi)(\theta) = (\phi)(t+\theta)$ if $-r \leq t+\theta \prec 0$.

(b) The operator $\tilde{A}$ defined by (2) is an infinitesimal generator of $(\tilde{T}(t)$ (i.e. $\tilde{A} = B$).

Hence,

$$\frac{(\tilde{J}_{\frac{t}{n}}^{n}\phi)(0)-\phi(0)}{t} - \dot{\phi}(0) = \frac{(\tilde{J}_{\frac{t}{n}}^{n}\phi)(0)-\phi(0)}{t} - A\phi(0) - F(\phi). \qquad (10)$$

$= \frac{1}{n}\sum_{i=0}^{n-1} J_{\frac{t}{n}}^{(i+1)}[F(\tilde{J}_{\frac{t}{n}}^{n-i}\phi)$ - $F(\phi)] + \frac{1}{n}\sum_{i=0}^{n-1}[J_{\frac{t}{n}}^{(i+1)}$ - $I]$ $F(\phi)]$
$+ \frac{1}{t}$ [$[J_{\frac{t}{n}}^{n}$- I - $tA]\phi(0)$

And by lemma 2, we have
$\lim_{t\to 0}\frac{\tilde{T}(t)\phi)(0)-\phi(0)}{t} = \lim_{t\to 0}\ lim_{t\to+\infty}\frac{1}{t}[\ (\ \tilde{J}_{\frac{t}{n}}^{n}\phi)(0)$ - $\phi(0)]$

$= A\phi(0) + F(\phi)$

$= \dot{\phi}(0)$;

thus $\phi \in D(B)$ and $B\phi = \tilde{A}\phi = \dot{\phi}$.
Conversely, we shall prove that if $\phi \in D(B)$ then $\phi \in D(\tilde{A})$ and $\tilde{A}\phi = B\phi$.In fact, let $phi \in D(B)$, then $B\phi = \lim_{t\to 0}\frac{\tilde{T}(t)\phi)-\phi}{t} = \dot{\phi}$ existe in C, so,$\phi \in C^1$ and

$$\phi(0) = \lim_{t\to 0}\frac{(\tilde{T}(t)\phi)(0)-\phi(0)}{t} = \lim_{t\to 0}\ lim_{t\to+\infty}\frac{1}{t}[(\tilde{J}_{\frac{t}{n}}^{n}\phi)(0)-\phi(0)]. \qquad (11)$$

We will prove that $\dot{\phi}(0) = A\phi(0) + F(\phi)$. From (14) and lemma 2, we have $\lim_{t\to 0}\ lim_{t\to+\infty}\frac{1}{t}[\ (\ \tilde{J}_{\frac{t}{n}}^{n}\phi)(0)$ - $\phi(0)] = A\phi(0) + F(\phi)$.
So, by (11) one obtains $\dot{\phi}(0) = A\phi(0) + F(\phi)$. Hence, $\phi \in D(\tilde{A})$ and $\tilde{A}\phi = B\phi = \dot{\phi}$.

3 Main result

Theorem 2 *: Suppose That (H_1) and (H_2) are satisfied and ϕ be an element of $C([-r,0];X)$.Then, the equation (1) has a unique solution, continuous on $[-r,+\infty[$ and given by*

$$x(t) = \phi(t) \text{ if } t \in [-r, 0]$$

$$x(t) = (\tilde{T}(t)\phi)(0) \text{ if } t \in [0,+\infty[.$$

In addition if for all $t \in [0,+\infty[$, $x(t) \in D(A)$, and the map $t \to Ax(t)$ is continuous, then the solutin $x(t)$ is of class C^1 on $[-r,+\infty[$.

Lemme 3 *: Suppose That (H_1) and (H_2) are satisfied.Then, for each $\phi \in C$, $t \geq 0$ and for all $j \in \{1,...m\}$, $k \in \{0,...p-1\}$, we have*

$$\lim_{p\to+\infty}\ lim_{m\to+\infty} J_{\frac{t}{mp}}^{m(p-k)-j+1}[F(\tilde{J}_{\frac{t}{mp}}^{mk+j}\phi) - F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)] = 0. \qquad (12)$$

and for all $p \in N$, we have

$$lim_{m\to+\infty}[J_{\frac{t}{mp}}^{m(p-k)-j+1} - J_{\frac{t}{mp}}^{m(p-k)}]F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi) = 0. \qquad (13)$$

Lemme 4 *: Suppose That (H_1) and (H_2) are satisfied.Then, for each $\phi \in C$, $t \geq 0$, we have*

$\lim_{p\to+\infty}\ lim_{m\to+\infty}$ [$\frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m} J_{\frac{t}{mp}}^{m(p-k)-j+1}$ F ($\tilde{J}_{\frac{t}{mp}}^{j+mk}$) - $\frac{t}{p}\sum_{k=0}^{p-1} J_{\frac{t}{mp}}^{m(p-k)}$ F ($\tilde{J}_{\frac{t}{mp}}^{mk}$)] = 0.

proof : For each $\phi \in C$ and ≥ 0, we have
$\left|\frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m} J_{\frac{t}{mp}}^{m(p-k)-j+1}F(\tilde{J}_{\frac{t}{mp}}^{j+mk}\phi) - \frac{t}{p}\sum_{k=0}^{p-1} J_{\frac{t}{mp}}^{m(p-k)}F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)\right|$
$\leq \frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m}\left|J_{\frac{t}{mp}}^{m(p-k)-j+1}F(\tilde{J}_{\frac{t}{mp}}^{j+mk}\phi) - J_{\frac{t}{mp}}^{m(p-k)}F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)\right|$
$\leq \frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m}\left|J_{\frac{t}{mp}}^{m(p-k)-j+1}[F(\tilde{J}_{\frac{t}{mp}}^{j+mk}\phi) - F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)]\right| +$
$\frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m}\left|[J_{\frac{t}{mp}}^{m(p-k)-j+1} - J_{\frac{t}{mp}}^{m(p-k)}]F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)\right|$;
and the result follows from lemma 3.

proof. f theorem 2 :

We have to show that $\tilde{T}(t)\phi$ verifies the delay equations (1). We will look at the integral form

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s)F(x_s)ds,$$

that is,

$$(\tilde{T}(t)\phi)(0) = T(t)\phi(0) + \int_0^t T(t-s)F(\tilde{T}(s)\phi)ds.$$

w write (9) as

$$(\tilde{J}_{\frac{t}{n}}^{n}\phi)(0) - J_{\frac{t}{n}}^{n}\phi(0) = \frac{t}{n}\sum_{i=0}^{n-1} J_{\frac{t}{n}}^{i+1}F(\tilde{J}_{\frac{t}{n}}^{n-i}\phi); \qquad (14)$$

by a change of variable n-i = j, one obtains
$\frac{t}{n}\sum_{i=0}^{n-1} J_{\frac{t}{n}}^{i+1}$ F ($\tilde{J}_{\frac{t}{n}}^{n-i}\phi) = \frac{t}{n}\sum_{j=1}^{n} J_{\frac{t}{n}}^{n-j+1}$ F ($\tilde{J}_{\frac{t}{n}}^{j}\phi)$,
and we change n by mp, we have

$$\frac{t}{mp}\sum_{j=1}^{mp} J_{\frac{t}{mp}}^{mp-j+1} F(\tilde{J}_{\frac{t}{mp}}^{j}\phi) = \frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m} J_{\frac{t}{mp}}^{m(p-k)-j+1} F(\tilde{J}_{\frac{t}{mp}}^{mk+j}\phi);$$

Also, from (4), we have

$$\frac{t}{p}\sum_{k=0}^{p-1} J_{\frac{t}{mp}}^{m(p-k)}F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi) \to \frac{t}{p}\sum_{k=0}^{p-1} T(t-\frac{k}{p}t)F(\tilde{T}(\frac{tk}{p})\phi), asm \to +\infty, \qquad (15)$$

$$\frac{t}{p}\sum_{k=0}^{p-1} T(t-\frac{k}{p}t)F(\tilde{T}(\frac{tk}{p})\phi), \to \int_0^t T(t-s)F(\tilde{T}(s)\phi)ds, asp \to +\infty \qquad (16)$$

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$$\left|\tfrac{t}{mp}\textstyle\sum_{k=0}^{p-1}\sum_{j=1}^{m} J_{\frac{t}{mp}}^{m(p-k)-j+1}F(\tilde{J}_{\frac{t}{mp}}^{mk+j}\phi)-\int_0^t T(t-s)F(\tilde{T}(s)\phi)ds\right|$$

$$\leq\left|\frac{t}{mp}\sum_{k=0}^{p-1}\sum_{j=1}^{m} J_{\frac{t}{mp}}^{m(p-k)-j+1}F(\tilde{J}_{\frac{t}{mp}}^{mk+j}\phi)-\frac{t}{p}\sum_{k=0}^{p-1} J_{\frac{t}{mp}}^{m(p-k)}F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)\right| \qquad (17)$$

$$+\left|\tfrac{t}{p}\textstyle\sum_{k=0}^{p-1} J_{\frac{t}{mp}}^{m(p-k)}F(\tilde{J}_{\frac{t}{mp}}^{mk}\phi)-\tfrac{t}{p}\sum_{k=0}^{p-1}T(t-\tfrac{k}{p}t)F(\tilde{T}(\tfrac{tk}{p})\phi)\right|$$

$$+\left|\tfrac{t}{p}\textstyle\sum_{k=0}^{p-1}T(t-\tfrac{k}{p}t)F(\tilde{T}(\tfrac{tk}{p})\phi-\int_0^t T(t-s)F(\tilde{T}(s)\phi)ds\right|,$$

and from lemmas 3,4, (15) and (16), the right-hand side of (21) tends to zero as m,p $\to +\infty$. Therefore, if we pass to limit in (14) as n $\to +\infty$, we have

$$(\tilde{T}(t)\phi)(0) = T(t)\phi(0) + \int_0^t T(t-s)F(\tilde{T}(s)\phi)ds.$$

that is,

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s)F(x_s)ds, \qquad (18)$$

Now, we prove that x(t) is the unique solution of (1). In fact, from (H_1), (H_2) and (18), we have

$$\|x_t - y_t\| = sup_{-r\leq\theta\leq 0}|x(t+\theta)-y(t+\theta)|$$
$$= sup_{-r\leq\theta\leq 0}\left|\int_0^{t+\theta} T(t+\theta-s)(F(x_s)-F(y_s))ds\right|$$
$$\leq \alpha\int_0^t\|x_s-y_s\|\,ds;$$

So, by Gnonwall's lemma, we have $x_t = y_t$, for all t ≥ 0.

Theorem 3 *Suppose That (H_1) and (H_2) are satisfied.If we have ($\tilde{T}(t)\phi)(0)\in$ D(A) and t $\to$A($\tilde{T}(t)\phi)(0)$ is continuous on [0,+∞[.Then,*

(a) $\tilde{T}(t)\phi\in$ D(A), for all t $\geq$0.

(b) $\tilde{T}(t)C \subset D(\tilde{T})$, for all t $\geq$ r.

proof :(a) For all $\theta\in$[-r,0], we have $x_t = \tilde{T}(t)\phi$.i.e.

$$x(t+\theta) = (\tilde{T}(t)\phi)(\theta) = (\tilde{T}(t+\theta)\phi)(0) \text{ if } t+\theta\geq 0.$$

$$x(t+\theta) = (\tilde{T}(t)\phi)(\theta) = \phi(t+\theta) \text{ if } (t+\theta)\in[-r,0].$$

So,

$$(\tilde{T}(t)\phi)(\theta) = (\tilde{T}(t+\theta)\phi)(0) + \int_0^{t+\theta}T(t+\theta-s)F(\tilde{T}(s)\phi)ds \text{ if } t+\theta\geq$$

$$(\tilde{T}(t)\phi)(\theta) = \phi(t+\theta) \text{ if } (t+\theta)\in[-r,0]$$

We will prove that $\tilde{T}(t)\phi\in D(\tilde{A})$. In fact,

$$\tfrac{d}{d\theta}(\tilde{T}(t)\phi)(\theta) = A\,(\tilde{T}(t+\theta)\phi)(0) + F(\tilde{T}(t+\theta)\phi) \text{ if } t+\theta\geq 0.$$

$$\frac{d}{d\theta}(\tilde{T}(t)\phi)(\theta) = \frac{d}{d\theta}\phi(t+\theta) if (t+\theta)\in[-r,0].$$

$\phi\in D(\tilde{A})$, so $\phi\in C^1$ and $\dot{\phi} = A\phi(0) + F(\phi)$.Thus the map $\theta\to\frac{d}{d\theta}(\tilde{T}($ is continuous on [-r,0] and $\frac{d}{d\theta}(\tilde{T}(t)\phi)(0)=A(\tilde{T}(t)\phi(0)) + F(\tilde{T}(t)\phi)$.Tl $\tilde{T}(t)\phi\in$ D(A).

(b) Let $\phi\in$ C and t $\geq$ r(t+$\theta\geq$0). From (19), we have

$$\tfrac{d}{d\theta}(\tilde{T}(t)\phi)(\theta) = A\,(\tilde{T}(t+\theta)\phi)(0) + F(\tilde{T}(t+\theta)\phi)$$

As the same that (a) we have $\tilde{T}(t)\phi\in D(\tilde{A})$, for all $\phi\in$ C and t $\geq$

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